YITP-SB-07-15, BRX-TH-587.

Exact $N = 4$ correlators of AdS$_3$/CFT$_2$

Ari Pakman*
C.N.Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794-3840, USA

Amit Sever†
Brandeis Theory Group, Martin Fisher School of Physics,
Brandeis University, Waltham, MA, 02454-9110, USA

We extend to chiral $N = 4$ operators the holographic agreement recently found between correlators of the symmetric orbifold of M^4 at large N and type IIB strings propagating in $AdS_3 \times S^3 \times M^4$, where $M^4 = T^4$ or $K3$. We also present expressions for some bulk correlators not yet computed in the boundary.

I. INTRODUCTION

One of the simplest realizations of the AdS_{n+1}/CFT_n duality [1, 2, 3, 4] is the duality between type IIB string theory in $AdS_3 \times S^3 \times M^4$, where M^4 is either a torus T^4 or a $K3$ surface, and a two-dimensional $N = 4$ superconformal field theory in the moduli space of the symmetric product of M^4. This duality can be derived by considering the near horizon of a system of Q_1 D1 branes and Q_5 D5 branes wrapping M^4.

The bulk and the boundary theories have equivalent moduli spaces [5, 6], and on both sides of the duality there are special points where the theory has a solvable description. In the bulk, the special point corresponds to a supergravity frame without RR flux [7], where the string worldsheet is described, for Euclidean AdS_3, by H_3^+ and $SU(2)$ WZW models at level $k = Q_5$. The second special point corresponds in the boundary to the symmetric orbifold of $N = Q_1 Q_5$ copies of M^4.

Recently, progress was made in checking the duality of the two theories at the dynamical level [8, 9] by comparing correlators at these solvable points. It was shown there that the large N limit of certain three-point functions of chiral fields computed earlier in the symmetric product CFT agree precisely with string theory three-point functions computed in the sphere. This verification of the AdS_3/CFT_2 duality is surprising because the computations are carried out at different points in the moduli space, thus suggesting a non-renormalization theorem.

In [8] it was shown that computations in the bulk reproduce one of the correlators of chiral $SU(2)$ multiplets computed in the boundary in [10, 11]. In [9] it was shown that, for $M^4 = T^4$, the fusion rules and the structure constants of the complete $N = 2$ chiral ring in the bulk are in precise agreement with the boundary results of [12].

In this note we show that a simple computation allows to extend the comparison to those cases not considered in [8, 9]. We will show that the agreement of correlators for chiral $N = 4$ multiplets holds for all the boundary correlators computed in [11]. In addition, we will give expressions for three-point functions in the bulk for $M^4 = T^4$ which were not yet computed in the boundary.

*ari.pakman@stonybrook.edu
†asever@brandeis.edu

II. BULK-BOUNDARY AGREEMENT

Chiral $SU(2)$ multiplets in $AdS_3 \times S^3 \times T^4$ are operators satisfying $H = J$, where J is the $SU(2)$ spin and H is the $SL(2,R)$ spin, which is interpreted as the conformal dimension in the dual theory. Physical string operators of this kind are given, in the holomorphic sector, by three families [13], as shown in the table below.

Field	$H = J$	Range of H	Sector
$\mathbb{O}_h^0$	$h - 1 = j$	$0, 1/2 \ldots \frac{k-2}{2}$	NS
$\mathbb{O}_h^a$	$h - 1/2 = j + 1/2$	$1/2, 1 \ldots \frac{k-1}{2}$	R
$\mathbb{O}_h^2$	$h = j + 1$	$1, 3/2 \ldots k/2$	NS

Here $a = 1, 2$ correspond to the two holomorphic one-forms in T^4. The numbers h, j are the spins of the operators under the bosonic $SL(2,R)_{k+2}$ and $SU(2)_{k-2}$ which appear in the decomposition of the supersymmetric WZW models into bosonic WZW models and free fermions (see [9] for details). The number h takes the $k-1$ values $h = 1, 3/2 \ldots k/2$. For a given h, each operator has also an anti-holomorphic label, so the full operators are, e.g., $\mathbb{O}_h^{(0,2)}$, etc.

The same families of operators appear in the boundary theory [12], but the range of H there is larger. It is expected that additional operators in the bulk come from including spectrally flowed sectors of $SL(2,R)$ [14, 15, 16], which we will not consider here.

The operators depend on the variables $x, \bar{x}$, which are interpreted as the local variables of the boundary theory, and on $y, \bar{y}$, which are isospin $SU(2)$ variables [17]. They

are normalized as

$$\langle \mathbb{O}_h^{(\alpha,\bar\alpha)} \mathbb{O}_h^{(\alpha,\bar\alpha)} \rangle = \frac{(y_1-y_2)^{2J}(\bar y_1-\bar y_2)^{2\bar J}}{(x_1-x_2)^{2H}(\bar x_1-\bar x_2)^{2\bar H}}\,, \tag{1}$$

and can be expanded into modes with definite $J_0^3, \bar J_0^3$ eigenvalues,

$$\begin{aligned}\mathbb{O}_h^{(\alpha,\bar\alpha)}(y,\bar y) &= \sum_{M=-J}^{J}\sum_{\bar M=-\bar J}^{\bar J}\left(c_M^J c_{\bar M}^{\bar J}\right)^{1/2} \\ &\times\; y^{-M+J}\bar y^{-\bar M+\bar J}\mathbb{V}_{h,M,\bar M}^{(\alpha,\bar\alpha)}\,,\end{aligned} \tag{2}$$

where

$$c_M^J = \begin{pmatrix} 2J \\ M+J\end{pmatrix} = \frac{(2J)!}{(J+M)!(J-M)!}\,. \tag{3}$$

The modes $\mathbb{V}_{h,M,\bar M}$ are normalized as

$$\langle \mathbb{V}_{h,M,\bar M}^{(\alpha,\bar\alpha)} \mathbb{V}_{h,-M,-\bar M}^{(\alpha,\bar\alpha)} \rangle = (-1)^{J+\bar J-M-\bar M}\,, \tag{4}$$

where we have taken $x_1=\bar x_1=1, x_2=\bar x_2=0$.

The string theory three-point functions for chiral operators were shown in [9] to be

$$\begin{aligned}\langle \mathbb{O}_{h_1}^{(\alpha_1,\bar\alpha_1)} \mathbb{O}_{h_2}^{(\alpha_2,\bar\alpha_2)} \mathbb{O}_{h_3}^{(\alpha_3,\bar\alpha_3)} \rangle &= \frac{N^{-\frac12} f(h_i;\alpha_i) f(h_i;\bar\alpha_i)}{\sqrt{(2h_1-1)(2h_2-1)(2h_3-1)}} \\ &\times y_{12}^{J_1+J_2-J_3} y_{23}^{J_2+J_3-J_1} y_{31}^{J_3+J_1-J_2} \\ &\times \bar y_{12}^{\bar J_1+\bar J_2-\bar J_3} \bar y_{23}^{\bar J_2+\bar J_3-\bar J_1} \bar y_{31}^{\bar J_3+\bar J_1-\bar J_2}\,,\end{aligned} \tag{5}$$

where $y_{12}=y_1-y_2$, etc., the operators are at $x=0,1,\infty$, and the functions $f(h_i;\alpha_i)=f(h_i;\alpha_1,\alpha_2,\alpha_3)$ are given by

$$\begin{aligned} f(h_i;0,0,0) &= -h_1-h_2-h_3+2 \\ f(h_i;0,0,2) &= -h_1-h_2+h_3+1 \\ f(h_i;0,2,2) &= -h_1+h_2+h_3 \\ f(h_i;2,2,2) &= h_1+h_2+h_3-1 \\ f(h_i;0,a,b) &= f(h_i;2,a,b) \\ &= \sqrt{(2h_2-1)(2h_3-1)}\,\xi^{ab}\,,\end{aligned} \tag{6}$$

with $\xi^{12}=\xi^{21}=1, \xi^{11}=\xi^{22}=0$. Note that all the dependence on the type of operator $\alpha_i,\bar\alpha_i$ is encoded in the functions $f(h_i;\alpha_i)$ and is completely factorized in (5) between holomorphic and anti-holomorphic sectors.

In [9] only $N=2$ chiral states were considered, so the relation $J_3=J_2+J_1$ was imposed and only the $M_{1,2}=J_{1,2}$, $M_3=-J_3$ members of the $SU(2)$ multiplet were retained by taking the limits $y_{1,2}\to 0, y_3\to\infty$. Here we will keep both J_i's and y_i's arbitrary, the only restriction coming from the $SU(2)$ fusion rules applied to the j_i's and U(1) R-charge conservation $M_1+M_2+M_3=0$ (and similarly for the $\bar M_i$'s). This case was considered in [8] for correlators with $\alpha_i=\bar\alpha_i=0$, and $M=\bar M$. In this note, we consider arbitrary $\alpha_i,\bar\alpha_i=0,2$ and $M,\bar M$. Our results will thus be valid for both $M^4=T^4$ and $M^4=K3$, since only for operators with $\alpha,\bar\alpha=a$ these two cases differ. Correlators involving $\alpha,\bar\alpha=a$ $N=4$ chiral primaries with $J_3<J_1+J_2$ were not computed yet in the boundary conformal field theory, so for these cases we will present the predictions from the bulk for $M^4=T^4$.

Let us express the operators in terms of

$$n = 2h-1\,, \tag{7}$$

where, in the symmetric orbifold, n is the length of the permutation cycle in the corresponding operator. Let us label also the two types of operators by $\epsilon=-1$ for $\alpha=0$ and $\epsilon=+1$ for $\alpha=2$. The spins are given now by

$$J_i = \frac{n_i+\epsilon_i}{2} \qquad \bar J_i = \frac{n_i+\bar\epsilon_i}{2}\,. \tag{8}$$

Remarkably, all the correlators with $\alpha=0,2$, which were computed in [9] separately for each case, can be expressed in terms of n_i,ϵ_i in a symmetric form as

$$\begin{aligned}&\langle \mathbb{O}_{n_1}^{(\epsilon_1,\bar\epsilon_1)} \mathbb{O}_{n_2}^{(\epsilon_2,\bar\epsilon_2)} \mathbb{O}_{n_3}^{(\epsilon_3,\bar\epsilon_3)} \rangle = \\ &\frac{1}{\sqrt N}\frac{(\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3+1)(\bar\epsilon_1 n_1+\bar\epsilon_2 n_2+\bar\epsilon_3 n_3+1)}{4(n_1n_2n_3)^{1/2}} \\ &\times y_{12}^{J_1+J_2-J_3} y_{23}^{J_2+J_3-J_1} y_{31}^{J_3+J_1-J_2} \\ &\times \bar y_{12}^{\bar J_1+\bar J_2-\bar J_3} \bar y_{23}^{\bar J_2+\bar J_3-\bar J_1} \bar y_{31}^{\bar J_3+\bar J_1-\bar J_2}\,.\end{aligned} \tag{9}$$

To compare with the results of [11] we should recast this expression in the $M,\bar M$ basis. Expanding (9) using (2), it is easy to read out the term

$$\begin{aligned}&\langle \mathbb{V}_{n_1,-J_1,-\bar J_1}^{(\epsilon_1,\bar\epsilon_1)} \mathbb{V}_{n_2,J_2,\bar J_2}^{(\epsilon_2,\bar\epsilon_2)} \mathbb{V}_{n_3,J_1-J_2,\bar J_1-\bar J_2}^{(\epsilon_3,\bar\epsilon_3)} \rangle = \\ &\frac{N^{-\frac12}}{\left(c_{J_1-J_2}^{J_3} c_{\bar J_1-\bar J_2}^{\bar J_3}\right)^{1/2}} \frac{(\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3+1)(\bar\epsilon_1 n_1+\bar\epsilon_2 n_2+\bar\epsilon_3 n_3+1)}{4(n_1n_2n_3)^{1/2}}\,,\end{aligned} \tag{10}$$

where we have used $(-1)^{2(J_3+\bar J_3)}=1$, as follows from (8). The general correlator in the $M,\bar M$ basis follows from the Wigner-Eckart theorem and is given by

$$\begin{aligned}&\langle \mathbb{V}_{n_1,M_1,\bar M_1}^{(\epsilon_1,\bar\epsilon_1)} \mathbb{V}_{n_2,M_2,\bar M_2}^{(\epsilon_2,\bar\epsilon_2)} \mathbb{V}_{n_3,M_3,\bar M_3}^{(\epsilon_3,\bar\epsilon_3)} \rangle = \\ &\langle \mathbb{V}_{n_1,-J_1,-\bar J_1}^{(\epsilon_1,\bar\epsilon_1)} \mathbb{V}_{n_2,J_2,\bar J_2}^{(\epsilon_2,\bar\epsilon_2)} \mathbb{V}_{n_3,J_1-J_2,\bar J_1-\bar J_2}^{(\epsilon_3,\bar\epsilon_3)} \rangle \\ &\times \frac{d_{M_1,M_2,M_3}^{J_1,J_2,J_3} d_{\bar M_1,\bar M_2,\bar M_3}^{\bar J_1,\bar J_2,\bar J_3}}{d_{-J_1,J_2,J_1-J_2}^{J_1,J_2,J_3} d_{-\bar J_1,\bar J_2,\bar J_1-\bar J_2}^{\bar J_1,\bar J_2,\bar J_3}}\,,\end{aligned} \tag{11}$$

where

$$d_{M_1,M_2,M_3}^{J_1,J_2,J_3} = \begin{pmatrix} J_1 & J_2 & J_3 \\ M_1 & M_2 & M_3\end{pmatrix} \tag{12}$$

are the $SU(2)$ 3j symbols. Using now

$$d_{-J_1,J_2,J_1-J_2}^{J_1,J_2,J_3} = \left[\frac{(2J_1)!(2J_2)!}{(J_2+J_2-J_3)!(J_1+J_2+J_3+1)!}\right]^{1/2} \tag{13}$$

we get

$$\begin{aligned}&\langle \mathbb{V}_{n_1,M_1,\bar M_1}^{(\epsilon_1,\bar\epsilon_1)} \mathbb{V}_{n_2,M_2,\bar M_2}^{(\epsilon_2,\bar\epsilon_2)} \mathbb{V}_{n_3,M_3,\bar M_3}^{(\epsilon_3,\bar\epsilon_3)} \rangle = \\ &L(J_i,M_i)L(\bar J_i,\bar M_i) \\ &\times\; \frac{1}{\sqrt N}\frac{(\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3+1)(\bar\epsilon_1 n_1+\bar\epsilon_2 n_2+\bar\epsilon_3 n_3+1)}{4(n_1n_2n_3)^{1/2}}\end{aligned} \tag{14}$$

where

$$L(J_i, M_i) = d^{J_1,J_2,J_3}_{M_1,M_2,M_3} \tag{15}$$
$$\times \left[\frac{(J_1+J_2-J_3)!(J_2+J_3-J_1)!(J_3+J_1-J_2)!(J_1+J_2+J_3+1)!}{(2J_1)!(2J_2)!(2J_3)!}\right]^{1/2} .$$

Eq.(14), which is the main result of this note, coincides precisely with eq.(6.47) of [11], with the identifications $n_1=n, n_2=m, n_3=q, \epsilon_1=1_n, \epsilon_2=1_m, \epsilon_3=1_q$.

Correlators involving operators with $\alpha, \bar{\alpha}=a$ are expressed similarly in the $M, \bar{M}$ basis using (6). There are essentially three classes of such correlators, given by

$$\langle \mathbb{V}^{(a,\bar{\epsilon}_1)}_{n_1,M_1,\bar{M}_1} \mathbb{V}^{(b,\bar{\epsilon}_2)}_{n_2,M_2,\bar{M}_2} \mathbb{V}^{(\epsilon_3,\bar{\epsilon}_3)}_{n_3,M_3,\bar{M}_3} \rangle = \tag{16}$$
$$L(J_i, M_i)L(\bar{J}_i, \bar{M}_i) \times \frac{1}{\sqrt{N}} \frac{\xi^{ab}(\bar{\epsilon}_1 n_1+\bar{\epsilon}_2 n_2+\bar{\epsilon}_3 n_3+1)}{2(n_3)^{1/2}} ,$$
$$\langle \mathbb{V}^{(a,\bar{a})}_{n_1,M_1,\bar{M}_1} \mathbb{V}^{(b,\bar{b})}_{n_2,M_2,\bar{M}_2} \mathbb{V}^{(\epsilon_3,\bar{\epsilon}_3)}_{n_3,M_3,\bar{M}_3} \rangle = \tag{17}$$
$$L(J_i, M_i)L(\bar{J}_i, \bar{M}_i) \times \frac{1}{\sqrt{N}} \xi^{ab}\xi^{\bar{a}\bar{b}} \left(\frac{n_1 n_2}{n_3}\right)^{1/2} ,$$
$$\langle \mathbb{V}^{(a,\bar{\epsilon}_1)}_{n_1,M_1,\bar{M}_1} \mathbb{V}^{(b,\bar{b})}_{n_2,M_2,\bar{M}_2} \mathbb{V}^{(\epsilon_3,\bar{a})}_{n_3,M_3,\bar{M}_3} \rangle = \tag{18}$$
$$L(J_i, M_i)L(\bar{J}_i, \bar{M}_i) \times \frac{1}{\sqrt{N}} \xi^{ab}\xi^{\bar{a}\bar{b}} (n_2)^{1/2} .$$

It would be interesting to extend the computations of [11] in order to verify the holographic agreement for these correlators.

III. DISCUSSION

The bulk-boundary agreement found is surprising because the computations are done at two largely separated points in the moduli space, suggesting a non-renormalization theorem which should be investigated. Since the agreement found here is valid at large N, the question arises whether such non-renormalization theorem would hold also at finite N, and if so, how the finite N corrections should be obtained in the bulk [18]. Among other interesting open questions, in [9] it was pointed out that for chiral operators in the boundary there are several ways of combining the fermions which multiply the twist fields. It would be interesting to understand what these options correspond to in the bulk.

Acknowledgements

We thank Oleg Lunin for conversations that led to this work and David Kutasov, Albion Lawrence, William Linch and John McGreevy for useful discussions. AP thanks the University of Chicago and AS thanks the MIT Center for Theoretical Physics for their generous hospitality. The work of AP is supported by the Simons Foundation. The work of AS is supported in part by the National Science Foundation under Grant PHY-0331516, by DOE Grant No. DE-FG02-92ER40706 and by an Outstanding Junior Investigator award.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. **B428**, 105 (1998), hep-th/9802109.
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.
[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.
[5] R. Dijkgraaf, Nucl. Phys. **B543**, 545 (1999), hep-th/9810210.
[6] F. Larsen and E. J. Martinec, JHEP **06**, 019 (1999), hep-th/9905064.
[7] J. M. Maldacena and A. Strominger, JHEP **12**, 005 (1998), hep-th/9804085.
[8] M. R. Gaberdiel and I. Kirsch (2007), hep-th/0703001.
[9] A. Dabholkar and A. Pakman (2007), hep-th/0703022.
[10] O. Lunin and S. D. Mathur, Commun. Math. Phys. **219**, 399 (2001), hep-th/0006196.
[11] O. Lunin and S. D. Mathur, Commun. Math. Phys. **227**, 385 (2002), hep-th/0103169.
[12] A. Jevicki, M. Mihailescu, and S. Ramgoolam, Nucl. Phys. **B577**, 47 (2000), hep-th/9907144.
[13] D. Kutasov, F. Larsen, and R. G. Leigh, Nucl. Phys. **B550**, 183 (1999), hep-th/9812027.
[14] J. M. Maldacena and H. Ooguri, J. Math. Phys. **42**, 2929 (2001), hep-th/0001053.
[15] Y. Hikida, K. Hosomichi, and Y. Sugawara, Nucl. Phys. **B589**, 134 (2000), hep-th/0005065.
[16] R. Argurio, A. Giveon, and A. Shomer, JHEP **12**, 003 (2000), hep-th/0009242.
[17] A. B. Zamolodchikov and V. A. Fateev, Sov. J. Nucl. Phys. **43**, 657 (1986).
[18] Note that in the boundary, these finite N corrections come from a covering space with the topology of a sphere.